Envision Algebra 1

A very quick review of the *Envision algebra one* book that I chose because it was on your list of better algebra books and I have often seen it used and it really is (sadly) one of the better choices out there.

I would very much prefer to work with the *University of Chicago School Mathematics Project*, *Algebra* but it was not reviewed by these sites at all. The U of C book is vastly better and really does agree with much of my critique as can be seen by the little emphasis it places on *factoring quadratics* for example. There is a bad mistake in its "proof" of *rational roots* and *integer factorability* being logically equivalent-but at least it even attempts to go there! Of course, even this actually great math book does *nothing* to call out the vast universe of math *confusions* that haunt school-math reality and torment students every day. That job is for the WEHM Guide!

Absolute value inequalities

On p.45 we have a very good example of an actual mistake in math education. *Absolute value inequalities*. For years I could not understand why this particular corner of the absolute value topic consistently remains an issue for many students, even good students. Finally, I figured out where it was coming from. As I often find with my students, the cause of the confusion can be neatly traced back to one practice.

When solving a *basic absolute value equation*, the first step is to divide it into two equations:

$$|x-3| = 1$$

$$x - 3 = 1 \; and \; -(x-3) = 1$$

Because that's what absolute value really means. If the content of the brackets is positive you leave it alone and if it is negative, you negate the whole thing (making it positive.)

Notice that the second part could also conveniently (why is this more convenient, really?) be re-written as:

$$(x-3) = -1$$

This is what is immediately done in school and in this book. It does, in a way, make sense of course, but as we say in chess: it is inexact. The problem school-math has now caused is exposed immediately in the next topic to come along *absolute value inequalities.*

$$|x-3| < 1$$

Now what? Well, take a guess. Even in this fine book we have the following non-sequitur causing problems for years to come. This inequality is again divided up but now something weird happens:

$$x - 3 < 1 \; and \; x - 3 > -1$$

Not only is the 1 (the left equation) now negative but the inequality has also suddenly, *inexplicably* flipped. There is no explanation in this (better) book or ever in school-math. Students must either be extremely curious and talented with a great background to figure out themselves what lies behind this mysterious step (approx. 1 % of students) or they silently give up and memorize (99%). Most students really have no choice but to blindly repeat this, to them, non-sensical pattern. I have even had teachers penalize students for doing it correctly after I taught them the following.

This could all be avoided of course by not being cute and inexact in the first place. It would also be a good way to practice handling inequalities and negatives, always also an issue.

$$|x-3|<1$$

$$x-3<1 \qquad \text{and} \qquad -(x-3)<-1$$

$$x-3<1 \qquad\qquad \text{and } then \qquad\qquad (x-3)>-1.$$

$$\text{(multiplying by a minus flips the inequality)}$$

Point-slope form

On p. 63 there is a pretty good[1] example of why implementing these higher quality textbooks alone never moved the needle in math scores. *Point-slope form* is given its own chapter as if it were nearly as important conceptually for 8th graders as *slope-intercept form*.

A quick review:

Slope-intercept form: Written as what 8th graders have so far been told a *function* looks like, a machine that produces *y*s. This form clearly shows the slope *m* and the y-intercept *b*.

Point-slope form: algebraic manipulation thereof that no longer starts off with *y*=, thereby obscuring the parameters *slope* and *y-intercept* and of course no longer looking anything like a *function*. The purpose of this confusing looking formula is to *get an equation from a point and a slope* that must then be rearranged into the *slope-intercept form*. This "avoids" having to find the *b* using information (point is on the line) from the *slope-intercept form*.

But it is fundamentally important to be able to start "from scratch" from the *slope-intercept form* and *two points* (or a *point* and a *slope*)! This is the first case of the unspeakably important concept of using information (*point* is on the line) to find unknowns (*b*, the *y-intercept*). This concept becomes more and more central to mathematics the further you progress. This starts immediately next year in 9th grade when finding a *quadratic* from *3 points*, or the *vertex* and *one* other *point* (there is, to wit, no point-quadratic form.) This idea is in fact, possibly, the very essence of mathematics, and also a mathematical concept that can be explicitly viewed as actual wisdom (what is it you want to know: #unknowns? what is it you know: #pieces of information?) I cannot even guess at the number of

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[1] I can see many education professionals defending point-slope form to the death. Teachers will claim it strengthens algebra, which it most definitely does not. The algebra steps that are needed to get this form back to slope-intercept form are exactly the same every single time; so the poor students just memorize them as well. The sole "advantage" is not having to *think* about how a function works: "I have a point I can plug into a function to find an unknown".
Mathematicians will tell you *point-slope* does, in fact, represent a more fundamental form of a line (even if it does not look like a function anymore) because a general point and a slope are more important than some arbitrary specific point (y-intercept), but this does not become a consideration until multivariable calculus (when a function is just as important as the idea of a manifold) and is wildly out of place in 8th grade during the introduction of the concept of a function. The same is true of the next chapter *standard form*.

students I have had that definitely did not understand how to do this later (all the way to college), precisely because the *point-slope form* avoided it for them from the beginning.

In short, if you really do understand *slope-intercept form* and how to find *b* from a point on the graph it certainly won't kill you to use the *point-slope formula* over and over instead, but for everyone else it will obscure one of the most important processes in mathematics that absolutely needs to be understood at this early stage.

So instead of explaining what I just did to keep the teacher from repeating the confusion/rote memorization pattern he or she was taught decades ago, and helping students actually understand the purpose of math, this book simply states both forms in separate chapters as if they were of equal significance. Not only does this distorts the order of importance of concepts at this stage, but what it really means, in the reality of school-math, is that teachers will simply skip straight to the *point-slope form* that they themselves were forced to memorize and barely touch on finding *b* from the original function. I can tell you that this has been the case with at least 85% of my hundreds of students.

This represents a good example of how these better materials may not have outright mistakes and won't focus on illogical unnecessary stuff as much as the vast majority of school-math materials (see example video, sample chapters), but they *do not call them out* either. In fact, one could reasonably assume that including both of these forms on equal footing might be some form of political compromise. I wonder what the process would be of getting a schoolbook approved that actually goes against the grain of the usual course of school-math instead of just silently offering better ways but keeping the old habits alive and well? The Common Core certainly does not say use *point-slope form* in 8th grade when functions are being introduced! So, where did it come from? Why is it still there?

Standard form: This is also introducing functions in a non-function format as equations to be satisfied. At least *standard form* is not used to avoid a central concept (as is *point-slope form)* and the manipulations required to get this form back into function form are a good exercise because standard form is not always presented identically, and therefore requires a real understanding of algebra to be rearranged correctly. Also, in the best case, it establishes the connection between simultaneous equations and lines. Still, it does confuse most students about what a function is and needs to be very carefully explained, which it basically never is, certainly not in this book.

<u>Relations and functions p.89</u>

Now after all this, the book suddenly decides it's time to "explain" what a *function* is. A very odd order of topics since we were just not only working on linear functions (including their graphs!) but even, for some reason, rearranging them into unrecognizable forms (see above). At least this book does not use the ridiculous *vertical line test* (see example video) to "test" whether a *relation* is a *function* or not, but once again (as always in school-math) instead of actually using a simple example such as a bank account (you can't have two different amounts of money in one account at the same time) they launch into formal math garble[2] bound to confuse and turn off any 8/9th grader. Then on p.95 they *return* to linear functions. The main difference between this new chapter and the one about all the different forms on p.63 seems to be the f(x) notation. But this is not clearly explained, and so the vast majority of students are undoubtably left wondering what the difference between *linear functions* and *slope-intercept*, *point-slope*, *standard form* is. Probably they conclude that they are not the same thing, which they of course most definitely are!

The order of these sections should be: 1. What is a function, 2. Linear functions, 3. The different forms of a linear function (including, if they must, the confusing rearrangements that don't even look like functions anymore.)

An unmitigated catastrophe!

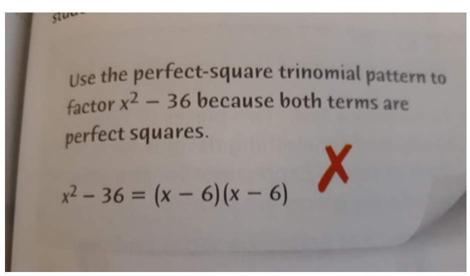
<u>p.281-312 factoring quadratics (31 pages)</u>

Here we go again with the single biggest catastrophe in all of school-math *quadratic factoring tricks*[3] (asides from possibly PEMDAS, see video, sample chapters.) 31 pages are devoted to factoring quadratic expressions by grouping etc. with no explanation of what they represent[4], why this is important or (God forbid!) interesting. Going into absurd detail of different cases such as: when *b* and *c* are positive, *b* is negative *c* is positive, *c* is negative p.291.

Quadratic factoring (especially the $a \neq 1$ $case$ $i.e$ $grouping$) along with completing the square is what, in reality, consumes the majority of school-math time in algebra 1 classrooms.

While wasting time on this long, useless and very boring detour the book uses traditional, confusing and misleading school-math terminology such as *monomial, binomial*, *trinomial* instead of *constant*, *linear*, *quadratic* (to wit, $3x^2 + 4x^2 + 7x^2 = 14x^2$ is a *trinomial* but also a *monomial*) which results in some real choice confusions such as this one on p.305:

How is anything involved here a *trinomial*??



[2] "*two or more elements of the domain of a function can map to one elements of the range, but two or more elements of the range cannot map to only one of the domain*" p.91

[3] Quadratic factoring tricks (as explained in the example video, sample chapters) do not apply to anything else in mathematics, do not represent a specific case of a more general principle and in the vast majority of cases do not supply a solution or even evidence of no solution. They are nothing more than useless knitting patterns to be memorized, a relic of the dark ages if there ever was one.

[4] Yes, there are a couple "real world" examples of how a quadratic may arise but let's be honest, nobody cares about these things until they can be graphed and visualized as functions, such as the path of a ball.

<u>Quadratic Functions p.312-396, 84 pages altogether.</u>

Breakdown of parts not about *quadratic functions* themselves:

344-354: linear exponential, quadratic models, 11 pages

363-376: quadratic factoring revisited, 13 pages

382-388: completing the square, 16 pages

So, overall from p. 281-396 we have:

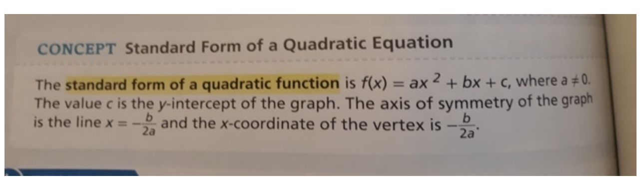
60 pages of quadratic factoring and completing the square

Only 44 pages are left for actual understanding the quadratic function: quadratic graphs, the vertex, the intercepts, the quadratic formula, the discriminant

Again, in school-math reality what happens is (as is reflected by the order and number of pages devoted in this book as well) *quadratic factoring* and *completing the square* is dwelled upon for many, many months while the actual working of the *quadratic function* itself is pretty much glossed over. Once again, this book though perhaps not guilty of the worst crimes (not clear actually), does nothing to correct the systemic problems in algebra 1 that have accumulated over the centuries!

Notable is also the order. Although the first time the student encounters *quadratics* is 31 pages of useless *factoring tricks* p.281-312, the *x-intercepts of a quadratic* (the whole reason we would even be interested in *factoring*) does not make its appearance until near the end of this whole incredible mess on p.363! In-between, the book for some reason opts to cover getting the *vertex* from the *vertex form* and graphing the *standard form without* knowing *the x-intercept's (??)*. This order (starting with the *vertex form* rather than the *standard form*) obscures the natural all important progression from a *linear function* $y = mx + b$ to a *quadratic function* $y = ax^2 + bx + c$, something that in my 25 years of experience students are always unnecessarily unclear about with catastrophic outcomes.

There is another odd little bow to the confusions of school-math on p.330. The *vertex formula* is introduced earlier here than in the usual course of school-math where it does not appear until well after *completing the square,* which is how, in school-math reality, students are mostly still taught to find the *vertex*, regardless of this books order (see example video, sample chapter). The book still somehow can't quite bring itself to say that the *axis of symmetry* and the *x-coordinate of the vertex* are indeed the exact same thing (or at least the exact same value) even though they have them right next to each other with identical formulas...it's just odd, especially knowing that in reality this is a major source of confusion for almost all students.


CONCEPT Standard Form of a Quadratic Equation

The **standard form of a quadratic function** is $f(x) = ax^2 + bx + c$, where $a \neq 0$. The value c is the y-intercept of the graph. The axis of symmetry of the graph is the line $x = -\frac{b}{2a}$ and the x-coordinate of the vertex is $-\frac{b}{2a}$.

Also, there is the odd insistence, perpetuated again here in this book, that the *c* in *standard form* is somehow important for understanding the *graph of a quadratic*, which it simply is not (the *y-intercept* tells us virtually nothing about the *shape of a quadratic*.) School-math always sets this up as a supposed similarity to *lines* (y=mx+b) where the *b* is indeed *fundamentally* important for the graph. Taking issue with this may seem petty or tedious here, but I can assure you that it is precisely these sorts of tiny *needle pricks of confusion* applied over and over again that leave students completely dazed and confused and afflicted with serious cases of *math trauma*. I am constantly reassuring my students that the *c* in a *quadratic* is not something they need to concern themselves with, but that they should rather concentrate on the *vertex* and the *quadratic formula* and the *discriminant* which are always systematically underserved topics.

I have not even addressed the most fundamental confusion which leads to real long-term conceptual damage, even for advanced students (I recently had to set a multivariable student straight about this.) *Factoring* is held up as an important, often as *the* most important, method of determining the *roots* of a *quadratic*, but not a word is wasted here on what it means if a quadratic does *not* factor nicely. This is actually a plus for this book because, at least, it does not introduce the insane term *prime quadratic* (*quadratics* that are not in *integer factorable*) which is unfortunately everywhere in the classrooms and textbooks of school-math algebra 1 (google it). But, once again, even though the book avoids this particular catastrophe (by just not mentioning it) it also does nothing to explain *why* it is a catastrophe. Students are, of course, still confronted by the term *prime quadratic* in school-math reality. The book also does nothing to actually clear up the question of what it means if a *quadratic* does *not* factor nicely.

When the *quadratic formula* is introduced the book does say "this will always give you the roots if there are any" but nowhere is it clearly stated that *factoring* will (most of the time!!) *not* give you the roots even if there are any. This may be mathematically obvious, but in the reality of school-math most students (again even top-level students) are never quite clear on this point and are often confused because it seems strange that school-math would dwell on *factoring* so much if this is indeed the case! This effect is amplified by the omnipresent term *prime quadratic* that would seem to mean something but (outside of group theory, see below) simply does not.

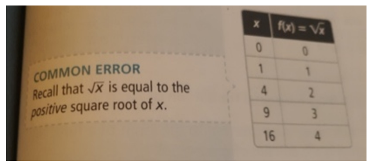
The surprising fact that as a result of group theory it can be said that having *rational roots* and being *integer factorable* is indeed logically equivalent would be the only mathematically interesting thing to say about all these completely useless *factoring tricks* (and *prime quadratics*) that take up so much time in school! But this is admittedly far too advanced to be explained at this point, although sometimes echoes of this fact will be found bouncing off the walls of school classrooms and textbooks in odd misinterpretations making everything exponentially even more confusing. This fact does, amazingly, lead to a way to avoid *quadratic factoring* altogether (the *discriminant* must be a *perfect square*!)

Another point (asides from not using the term *prime quadratics*) that this book can (kind of) be commended for is that it refuses to go down the path of *imaginary roots of a quadratic* which is really ridiculously confusing and out of place when trying to grasp quadratics and their graphs so: Yay! But, once again, simply ignoring something does not make it go away. Unfortunately, the Common Core, in one of its few actually grievous mistakes, includes *imaginary roots* here in the syllabus!! So, whether this book mentions them or not, students will still be confronted by this completely abstract notion that has nothing to do with the graph of a quadratic and will be very confused. The book will not help them at all here, but rather simply deserts them.

There are many smaller problematic details in the book that (because of the election and the due date of Nov 6) I will not get into such as:

p.363 the *zero-product property* deserves much more than just one line and at least one real example as it also is something students are consistently not quite as sure about as they should be!

p.411 the convention that the *square root function* only returns us the positive values is stated rather unsatisfyingly, in fact, incorrectly if you ask me (students are always confused about this):



Nowhere on this page does it say this is *only* true of the square root *function* not square roots *in general*. The word “recall” is odd as it seems to imply that this has always been the case which it obviously hasn’t (*quadratic formula!!*)

p.452 Always strange that the fact that only *strictly increasing* or *strictly decreasing* functions can have an *inverse* is not clearly stated as a way of understanding who has an inverse. Rather the concept of *one-to-one* is left as a completely abstract concept.